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Quantum electrodynamics with anisotropic scaling: Heisenberg-Euler action and Schwinger pair production in the bilayer graphene

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We discuss quantum electrodynamics emerging in the vacua with anisotropic scaling. Systems with anisotropic scaling were suggested by Hořava in relation to the quantum theory of gravity. In such vacua the space and time are not equivalent, and moreover they obey different scaling laws, called the anisotropic scaling. Such anisotropic scaling takes place for fermions in bilayer graphene, where if one neglects the trigonal warping effects the massless Dirac fermions have quadratic dispersion. This results in the anisotropic quantum electrodynamics, in which electric and magnetic fields obey different scaling laws. Here we discuss the Heisenberg-Euler action and Schwinger pair production in such anisotropic QED.

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1. INTRODUCTION

Both superfluid ^{3}He-A [1] and single-layer graphene [2, 3, 4] serve as examples of emerging relativistic quantum field theory in 3+1 and 2+1 dimensions, respectively. Both systems contain "relativistic" fermions, which are protected by the combined action of symmetry and topology (see review [5]), while the collective modes and deformations provide the effective gauge and gravity fields acting on these fermions [1, 3].

Effective electromagnetic field emerging in superfluid ^{3}He-A is the collective field which comes from the degrees of freedom corresponding to the shift of the position of the Weyl (conical) point in momentum space. The corresponding Maxwell action (for special orientation of the effective electric and magnetic fields, see details in [1]) is

$$S \sim \int d^3x dt \frac{v_F}{24\pi^2}\left[B^2 - \frac{1}{v_F^2}E^2\right] \ln \frac{1}{\left[B^2 - \frac{1}{v_F^2}E^2\right]}\,, \tag{1}$$

where v_F is the Fermi velocity. This action describes the effect of vacuum polarization, and corresponds to the Heisenberg-Euler action of standard quantum electrodynamics [6]. The only difference from the latter is that the fermions in the vacuum of ^{3}He-A are massless, and their masslessness is protected by momentum-space topology of the Weyl point. This results in the logarithmic term which describes the zero-charge effect – the screening of the effective charge. The effective action becomes imaginary when $E > v_F B$ describing Schwinger pair production in electric field [7] with the production rate $\propto E^2$ for the case of massless fermions [8] (Schwinger pair production of massive fermions has been discussed for the other topological superfluid – ^{3}He-B, see Ref. [9]). Similar action, but for the real electromagnetic field, should take place in the Weyl semimetals (for semimetals with the topologically protected Weyl points see Refs. [10, 11, 12, 13, 14, 15]). Among the other things, the massless fermions give rise to the chiral anomaly, and the corresponding Adler-Bell-Jackiw equation for anomaly has been verified in the ^{3}He-A experiments [16].

Graphene gives the opportunity not only to study the 2+1 quantum electrodynamics, but also extend the theory to different directions. In particular, to study systems with anisotropic scaling, which were suggested by Hořava for construction of the quantum theory of gravity, which does not suffer the ultraviolet (UV) divergencies (the UV completion of general relativity) [17, 18, 19, 20]. As distinct from the relativistic massless fermions in a single layer graphene, which obey the invariance under conventional scaling $\mathbf{r} \to b\mathbf{r}$, $t \to bt$, fermions in a bilayer ($N = 2$) or rhombohedral N-layer ($N > 2$) graphene have a smooth touching of the Dirac point, $E^2 \propto p^{2N}$, see Refs. [2, 4]. These fermions obey the anisotropic scaling $\mathbf{r} \to b\mathbf{r}$, $t \to b^N t$, precisely which is needed for construction of the divergence-free quantum gravity.

Here we discuss the effect of this anisotropic scaling on the effective action for real or artificial (e.g., created by deformations [3, 4]) electromagnetic fields. Due to anisotropic scaling, which distinguishes between the space and time components, the electric and magnetic fields experience different scaling laws. In par-

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ticular, the one-loop action for the magnetic field is $\propto B^2 p_0^{N-4+D}$, where D is the space dimension and p_0 is the infrared cut-off. This demonstrates that superfluid ^{3}He-A with its $D = 3$ and $N = 1$ and the bilayer graphene with its $D = N = 2$ both correspond to the critical dimension $D_c = 4 - N$ and thus they both give rise to the logarithmically divergent action for the magnetic field (which manifests itself as a logarithmic divergence of diamagnetic susceptibility in undoped bilayer graphene with the trigonal warping effects being neglected [22, 23]). However, for the electric field the action is $\propto E^2 p_0^{D-2-N}$. While for the superfluid ^{3}He-A this again gives the logarithmic divergence as it happens in the relativistic systems with massless fermions, in the multiple-layered graphene with its $D = 2$ the action diverges in the infrared for any $N > 1$, giving rise to the power-law action $\propto E^{(N+2)/(N+1)}$.

2. EFFECTIVE ACTION FOR REAL AND INDUCED ELECTROMAGNETIC FIELD

Anisotropic scaling for electromagnetic field

The effective Hamiltonian for the bilayer graphene in the simplest approximation is [2, 4]

$$\mathcal{H} = \frac{\sigma^+}{2m}\left((\hat{\mathbf{x}} + i\hat{\mathbf{y}})\cdot(\mathbf{p} - e\mathbf{A})\right)^2 + \frac{\sigma^-}{2m}\left((\hat{\mathbf{x}} - i\hat{\mathbf{y}})\cdot(\mathbf{p} - e\mathbf{A})\right)^2 \quad (2)$$

where σ are Pauli matrices and m is the mass entering the quadratic band touching. Experimentally, $m \approx 0.03 m_e$ where m_e is the free-electron mass [24]. Here we neglect the degrees of freedom related to the tetrad gravity and concentrate on the degrees of freedom corresponding to the electromagnetic field (in principle the vector potential $\mathbf{A}$ may include not only the real electromagnetic field, but also the collective field which come from the degrees of freedom corresponding to the shift of the position of the Dirac point in momentum space, as in ^{3}He-A). We shall use the natural units in which $\hbar = 1$; electric charge $e = 1$; the vector field $\mathbf{A}$ has dimension of momentum, $[A] = [L]^{-1}$; electric and magnetic fields have dimensions $[E] = [LT]^{-1}$ and $[B] = [L]^{-2}$, respectively.

For standard quantum electrodynamics emerging in the vacuum with massive electrons, the Lorentz invariance combined with the dimensional analysis gives the general form $(B^2 - E^2)f(x, y)$ for the Heisenberg-Euler action [6] in terms of dimensionless quantities $x = (B^2 - E^2)/M^4$, $y = \mathbf{B}\cdot\mathbf{E}/M^4$. Here M is the rest energy of electron, which violates the scale invariance in the infrared. Extension of the Born-Infeld electrodynamics to the anisotropic scaling has been considered in Ref. [25]. We consider the electrodynamics with anisotropic scaling, which is induced by fermions obeying the Hamiltonian (2).

General effective action

In Eq. (2) there is only one dimensional parameter, the mass m. Combining the dimensional analysis with the anisotropic scaling, one obtains that the effective action for the constant in space and time electromagnetic field, which is obtained by the integration over the 2+1 fermions with quadratic dispersion, is the function of the dimensionless combination μ of electric and magnetic field

$$S = \int d^2x dt\ \frac{B^2}{m} g(\mu) \quad , \quad \mu = \frac{m^2 E^2}{B^3}\,. \quad (3)$$

The change of the regime from electric-like to magnetic-like behavior of the action occurs at $\mu \approx 1$. The asymptotical behavior in two limit cases, $g(\mu \to 0) = a$ and $g(\mu \to \infty) = (b + ic)\mu^{2/3}$, gives the effective actions for the constant in space and time magnetic and electric fields:

$$S_B = a\int d^2x dt\ \frac{B^2}{m} \quad , \quad S_E = \int d^2x dt (b + ic) E^{4/3} m^{1/3}\,. \quad (4)$$

The dimensionless parameters a and b describe the vacuum polarization, and the dimensionless parameter c describes the instability of the vacuum with respect to the Schwinger pair production in the electric field, which leads to the imaginary part of the action.

This should be contrasted with the single layer graphene, where the corresponding effective action for the 2+1 relativistic quantum electrodynamics is

$$S \sim \int d^2x dt\ v_F \left[B^2 - \frac{1}{v_F^2}E^2\right]^{3/4} \quad , \quad (5)$$

critical field is $E_c(B) = v_F B$; and the Schwinger pair production is $\propto E^{3/2}$ at $B = 0$ (see Refs. [21, 26, 27]). Returning to the conventional units, one should substitute $B \to eB/c$ and $E \to eE$.

Action for magnetic field

First, let us consider the case $E = 0$. The action is

$$S = a\int d^2x dt \frac{B^2}{m} \quad (6)$$

where a is undefined yet numerical coefficient. The simplest way to find a is the following. First, let us consider the case of uniform magnetic field and the sample of unit area, than, $\int d^2x \to 1$. Second, let us go to the imaginary time, $\exp(iS) \to \exp(-\beta F)$ where F is the free energy, $\beta = 1/T$ is the inverse temperature. The change of the free energy in magnetic field is $F = -\chi B^2/2$

where χ is the magnetic susceptibility (per unit area). Thus, we have

$$a = \frac{m\chi}{2} \tag{7}$$

The susceptibility for the case of bilayer within the model of purely parabolic spectrum has been calculated in Refs. [22, 23]. As we know, the $D = 2$ vacuum of fermions with quadratic touching, $N = 2$, corresponds to the critical dimension for the magnetic field action: it is logarithmically divergent for the case of zero doping. This logarithm should be cut at the energy of the reconstruction of the spectrum due to the farther hopping effects ("trigonal warping") and/or interelectron interaction (for the most recent discussion, see Ref. [24]). The answer is

$$a = \frac{g_s g_v}{32\pi}\Lambda\,, \tag{8}$$

where $g_s = g_v = 2$ are spin and valley degeneracies. For small fields, the logarithmically running coupling is $\Lambda = \ln\left(\gamma_1^2/\Delta^2\right)$, where the ultraviolet cut-off is provided by the interlayer hopping γ_1, while the infrared cut-off is provided by the trigonal warping whose typical energy is $\Delta \approx 0.01\gamma_1$. For larger fields, $B > \Delta^2$, the infrared cut-off is provided by the field itself, $\Lambda = \ln\left(\gamma_1^2/B\right)$, which is similar to what occurs in effective electrodynamics in ^{3}He-A with massless fermions, see Eq.(1).

Schwinger pair production

Schwinger pair production in bilayer graphene in zero magnetic field has been considered in Ref. [27]. Let us consider the case of the crossed fields $\mathbf{B} = B\hat{\mathbf{z}}$, $\mathbf{E} = E\hat{\mathbf{x}}$ using the semiclassical approximation. We will use the gauge $A_x = 0, A_y = Bx$. Thus, the imaginary part of the momentum along x direction is determined by the equation (cf. Ref. [27]):

$$\kappa^2\left(x\right) = \left(k + Bx\right)^2 - 2mE\left|x\right| \tag{9}$$

where k is the (conserving) momentum in y-direction. The classically forbidden regions relevant for the tunneling is determined by the condition $\kappa^2\left(x\right) > 0$ and the tunneling exponent is determined by the imaginary part of the action

$$S_E\left(k\right) = 2\int\limits_{x_L}^{x_R} dx\kappa\left(x\right) \tag{10}$$

where x_L, x_R are the left and right turning points.

One has to introduce the parameter

$$\mathcal{E} = \frac{mE}{B} \tag{11}$$

and

$$X = k + Bx \tag{12}$$

thus,

$$S_E\left(k\right) = \frac{2}{B}\int\limits_{X_L}^{X_R} dX\kappa\left(X\right) \tag{13}$$

where

$$\kappa^2\left(X\right) = \begin{cases} \left(X - \mathcal{E}\right)^2 + \mathcal{E}\left(2k - \mathcal{E}\right), & X > k \\ \left(X + \mathcal{E}\right)^2 - \mathcal{E}\left(2k + \mathcal{E}\right), & X < k \end{cases} \tag{14}$$

Now we have to study conditions of existence of the left and right turning points. We can restrict ourselves by the case $k > 0$ only, due to symmetry with respect to the replacement $k \to -k, x \to -x$. Simple analysis show that two turning point exist only if

$$\mathcal{E} > 2k \tag{15}$$

The further calculations are straightforward, and the answer is

$$S_E\left(k\right) = \mu f\left(\frac{2k}{\mathcal{E}}\right) \tag{16}$$

where

$$f\left(x\right) = x - \frac{1+x}{2}\ln\left(1+x\right) - \frac{1-x}{2}\ln\left(\frac{1}{1-x}\right) \tag{17}$$

The Taylor expansion for the function f reads

$$f\left(x\right) = \sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n(2n+1)} \tag{18}$$

For $\mathcal{E} \to \infty$ we have $f\left(x\right) \approx x^3/6$, which is in agreement with Ref. [27], if one fixes the misprint in Eq.(36) of [27], where there is a power 1/3 instead of 3. The semiclassical approximation is valid if $\mu \gg 1$.

The pair production is obtained after integration of the tunneling exponent over the momentum k:

$$\begin{aligned} \mathrm{Im}\; S &= \frac{g_s g_v}{2\pi^2}E\int_0^{\mathcal{E}/2} dk\exp\left(-S_E\left(k\right)\right) \\ &= \frac{g_s g_v}{4\pi^2}B^2\mu\int_0^1 dx\exp\left(-\mu f(x)\right)\,. \end{aligned} \tag{19}$$

Semiclassically, there is *always* some states which demonstrate tunneling, but only with $|k| < \mathcal{E}/2$. In the case of strong magnetic field their contribution shrinks to the point.

In the limit of small magnetic fields one obtains the parameter c in the action (4):

$$\mathrm{Im}S = \frac{g_s g_v}{4\pi^2}B^2\mu\int_0^{+\infty} dx\exp\left(-\frac{\mu}{6}x^3\right) = cE^{4/3}m^{1/3}\,, \quad c = \frac{g_s g_v}{12\pi^2}6^{1/3}\Gamma(1/3). \tag{20}$$

The Schwinger pair production rate proportional to $E^{4/3}$ has been discussed in Ref. [27].

If we take into account the next-order corrections in $1/\mu$ the expression (20) is multiplied by the factor $1 - \frac{3^{5/3}2^{2/3}}{10\Gamma(1/3)}\frac{B^2}{(mE)^{4/3}}$.

The change of the regime from magnetic-like to electric-like happens at $\mu \approx 1$ which means, in CGSE units,

$$\frac{E}{B} \approx \frac{\hbar}{mc}\sqrt{\frac{|e|B}{\hbar c}} \tag{21}$$

For the field $B = 1$T this ratio is of the order of $3 \cdot 10^{-3}$, that is, the order of magnitude smaller than for the case of the single-layer graphene where it is $v_F/c \approx 1/300$.

Effective action for higher order touching

All this can be extended to the order of N band touching, which presumably may be achieved by rhombohedral stacking of N graphene-like layers [2, 4]. The effective Hamiltonian for fermions in this case is

$$\mathcal{H} = \frac{\sigma^+}{2m}\left((\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \cdot (\mathbf{p} - e\mathbf{A})\right)^N + \frac{\sigma^-}{2m}\left((\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \cdot (\mathbf{p} - e\mathbf{A})\right)^N . \tag{22}$$

If we are interested only in the infrared behavior of the action in the regime when the electric field dominates, then the induced electromagnetic action has the following general structure:

$$S(N) = \int d^2x dt \ \frac{B^{(2+N)/2}}{m} g(\mu) \ \ , \ \ \mu = \frac{m^2E^2}{B^{N+1}} . \tag{23}$$

At large μ one has $g(\mu) \propto \mu^{(N+2)/2(N+1)}$, and the Schwinger pair production rate in zero magnetic field is

$$\dot{n} \sim \frac{1}{m}(mE)^{\frac{N+2}{N+1}} . \tag{24}$$

3. DISCUSSION

In the condensed matter context (or in related microscopic theories of the quantum vacuum), the existence of the topologically protected nodes in spectrum of Weyl media gives rise to the effective gauge fields ($U(1)$ and $SU(2)$) and gravity. The same takes place in single layer and bilayer graphene, which contain Dirac points in their spectrum. The effective $SU(2)$ gauge field comes from the spin degrees of freedom, these are the collective modes in which the momentum of the Dirac point shifts differently for spin-up and spin-down species. In addition to the collective modes related to the shift of the Dirac point, the graphene has degrees of freedom, which correspond to tetrads in anisotropic gravity. In particular, for the bilayer graphene these degrees of freedom enter the effective Hamiltonian in the following way

$$\begin{aligned}\mathcal{H} = \sigma^+ \left((\mathbf{e}_1 + i\mathbf{e}_2) \cdot (\mathbf{p} - e\mathbf{A})\right)^2 \\ +\sigma^- \left((\mathbf{e}_1 - i\mathbf{e}_2) \cdot (\mathbf{p} - e\mathbf{A})\right)^2 ,\end{aligned} \tag{25}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ play the role of zweibein fields, which give rise to the energy spectrum $E^2 = \left(g^{ik}p_ip_k\right)^2$ corresponding to the effective 2D metric $g^{ik} = e^i_1e^k_1 + e^i_2e^k_2$; to spin connection and torsion.

As distinct from superfluid ^{3}He-A, which is the analog of the relativistic vacuum with massless Weyl fermions, the bilayer graphene is the representative of the quantum vacua, which experience different scaling laws for space and time. While such vacua were considered by Hořava in relation to quantum gravity, here we applied the anisotropic scaling to quantum electrodynamics emerging in these systems, using as an example the 2D system with massless Dirac fermions with quadratic spectrum. Such systems have peculiar properties, and we touched here the Heisenberg-Euler action and the Schwinger pair production.

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1. G.E. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford (2003).
2. A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, and A.K. Geim, Rev. Mod. Phys. **81**, 109–162 (2009).
3. M.A.H. Vozmediano, M.I. Katsnelson, and F. Guinea, Physics Reports **496**, 109–148 (2010).
4. M.I. Katsnelson, *Graphene: Carbon in Two Dimensions*, Cambridge Univ. Press, Cambridge (2012).
5. G.E. Volovik, Topology of quantum vacuum, draft for Chapter in proceedings of the Como Summer School on analogue gravity, arXiv:1111.4627.
6. W. Heisenberg and H. Euler, Z. Phys. **98**, 714–732 (1936).
7. J. Schwinger, Phys. Rev. **82**, 664–679 (1951).
8. G.E. Volovik, *Exotic properties of superfluid ^{3}He*, World Scientific, Singapore (1992).
9. N. Schopohl and G.E. Volovik, Ann. Phys. (N. Y.) **215**, 372–385 (1992).
10. A.A. Abrikosov and S.D. Beneslavskii, Sov. Phys. JETP **32**, 699–708 (1971).
11. H.B. Nielsen and M. Ninomiya, Phys. Lett. B **130**, 389–396 (1983).

12. A.A. Abrikosov, Phys. Rev. **B 58**, 2788–2794 (1998).
13. X. Wan, A.M. Turner, A. Vishwanath and S.Y. Savrasov, Phys. Rev. B **83**, 205101 (2011).
14. A.A. Burkov and L. Balents, Phys. Rev. Lett. **107**, 127205 (2011).
15. V. Aji, arXiv:1108.4426.
16. T.D.C. Bevan, A.J. Manninen, J.B. Cook, J.R. Hook, H.E. Hall, T. Vachaspati, and G.E. Volovik, Nature **386**, 689–692 (1997).
17. P. Hořava, Phys. Rev. Lett. **102**, 161301 (2009).
18. P. Hořava, Phys. Rev. D **79**, 084008 (2009).
19. P. Hořava, JHEP 0903, 020 (2009),
20. C. Xu and P. Hořava, Phys. Rev. D **81**, 104033 (2010).
21. J.O. Andersen and T. Haugset, Phys. Rev. D **51**,3073–3080 (1995).
22. S.A. Safran, Phys. Rev. B **30**, 421–423 (1984).
23. M. Koshino and T. Ando, Phys. Rev. B **76**, 085425 (2007).
24. A.S. Mayorov, D.C. Elias, M. Mucha-Kruczynski, R.V. Gorbachev, T. Tudorovskiy, A. Zhukov, S.V. Morozov, M.I. Katsnelson, V.I. Fal'ko, A.K. Geim, and K.S. Novoselov, Science **333**, 860–863 (2011).
25. O. Andreev, Int. J. Mod. Phys. A **25**, 2087–2101 (2010); arXiv:0910.1613.
26. D. Allor, T.D. Cohen, and D.A. McGady, Phys. Rev. D **78**, 096009 (2008).
27. N.M. Vildanov, J. Phys.: Condens. Matter **21**, 445802 (2009).